647510 **MSC3MatC302x** **Seat No :________**

M.Sc.(Maths.) Semester - 3 (*CBCS*) Examination
Nov/Dec-2023 (NEW COURSE)
Functional Analysis (Core (New))

Time: 2:30 Hours **Marks: 70**

Instructions:

1. Figures to the right indicate marks.
2. All questions are compulsory.

Que.1(A) Answer the following. (04)

1) Show that every finite-dimensional subspace of a normed space is complete.

Que.1(B) Answer any two questions out of three. (10)

1) Let X and Y be normed linear spaces and $X \neq \{0\}$. Then show that the space $B(X, Y)$ of all bounded linear operators from a normed space X into a normed space Y is a Banach space if and only if Y is a Banach space.
2) Let $C[a, b]$ denote the collection of all continuous functions $f: [a, b] \to \mathbb{R}$ for $1 \leq p \leq \infty$. Define $\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}$ and $\|f\|_\infty = sup\{|f(t)|: t \in [a, b]\}$. Then show that $\|\cdot\|_p$ is a norm on $C[a, b]$ for $1 \leq p \leq \infty$.
3) Let Y be a closed subspace of a normed space X. For $x + Y$ in the quotient space X/Y, let $\|x + Y\|_q = \inf\{\|x + y\|: y \in Y\}$.
Then show that a sequence $\{x_n + Y\}$ converges to $x + Y$ in X/Y if and only if there is a sequence $\{y_n\}$ in Y such that $\{x_n + y_n\}$ converges to x in X.

Que.2(A) Answer the following. (04)

1) Define equivalent norms. Show that all the norms on finite dimensional normed space are equivalent.

Que.2(B) Answer any two questions out of three. (10)

1) Let X be a normed space and Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Then show that there exists some $x_r \in X$ such that $\|x_r\| = 1$ and $r < dist(x_r, Y) \leq 1$.
2) Let X be a normed space. Then show that every closed and bounded subset of X is compact if and only if X is finite-dimensional.
3) Let $\{x_n\}$ be a weakly convergent sequence in a normed linear space X. Then show that
(i) weak limit x of $\{x_n\}$ is unique
(ii) every subsequence of $\{x_n\}$ converges weakly to x
(iii) the sequence $\{\|x_n\|\}$ is bounded

Que.3(A) Answer the following. (04)

1) Let X and Y be normed spaces and $F: X \to Y$ be a linear map. Then show that F is an open map if and only if there is $r > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq r\|y\|$.

Que.3(B) Answer any two questions out of three. (10)

1) State and prove uniform boundedness principle.
2) State and prove the closed graph theorem.
3) State and prove the Hahn-Banach extension theorem for normed linear space. State all results that you have used.

Que.4(A) Answer the following. (04)

1) State and prove Bessel's inequality for inner product space.

Que.4(B) Answer any two questions out of three. (10)

1) Let H be a Hilbert space and Y be a closed subspace of H. Then show that $Y \oplus Y^{\perp} = H$ and $Y^{\perp\perp} = Y$, where $(Y^{\perp})^{\perp} = Y^{\perp\perp}$.
2) State and prove the Riesz-Representation theorem for a Hilbert space.
3) Let $l^2 = \{x = (x(1), x(2), \dots): x(i) \in \mathbb{R} \; and \; \sum_{i=1}^{\infty} |x(i)|^2 < \infty\}$. For $x = (x(1), x(2), \dots)$ and $y = (y(1), y(2), \dots) \in l^2$, define $< x, y > = \sum_{n=1}^{\infty} x(n)\,\overline{y(n)}$. Then show that $<\cdot,\cdot>$ is an inner product on l^2.

Que.5(A) Answer the following. (04)

1) Let H be a Hilbert Space and $T: H \to H$ be a bounded linear operator. Then show that

   (i) T is isometry if and only if $T^*T = I$, where I is the identity operator on H

   (ii) T is unitary if and only if T is an onto isometry.

Que.5(B) Answer any two questions out of three. (10)

1) Let H_1 and H_2 be Hilbert spaces, $S: H_1 \to H_2$ and $T: H_1 \to H_2$ be bounded linear operators and α be any scalar. If S^* denotes the adjoint of S, then show that

   (i) $(S + T)^* = S^* + T^*$

   (ii) $(\alpha S)^* = \bar{\alpha} S^*$

   (iii) $(ST)^* = T^* S^*$

   (iv) $(S^*)^* = S$
2) Show that a bounded linear operator $P: H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and idempotent.
3) Define normal operator. Let H be a Hilbert space, $S: H \to H$ and $T: H \to H$ be bounded linear operators on H and T^* denotes the adjoint of T.

   (i) Show that T is normal if and only if $\|Tx\| = \|T^*x\|, \forall x \in H$

   (ii) Let S and T be normal. If S commutes with T^* and T commutes with S^*, then show that $S + T$ and ST are normal.

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